# Brane cosmological perturbations

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## Abstract

We address the question of cosmological perturbations in the context of brane cosmology, where our Universe is a three-brane where matter is confined, whereas gravity lives in a higher dimensional spacetime. The equations governing the bulk perturbations are computed in the case of a general warped universe. The results are then specialized to the case of a five-dimensional spacetime, scenario which has recently attracted a lot of attention. In this context, we decompose the perturbations into 'scalar', 'vector' and 'tensor' modes, which are familiar in the standard theory of cosmological perturbations. The junction conditions, which relate the metric perturbations to the matter perturbations in the brane, are then computed.


## I. INTRODUCTION

Recently has emerged the fascinating idea that extra dimensions (beyond our familiar time and three spatial dimensions) could be large today or at energy scales much below what was thought before [1]. The reason why these extra dimensions could remain hidden is that matter fields would be confined to a three-dimenional brane (our visible 'space') whereas gravitational fields would live everywhere and thus also in the extra-dimensions.

Even non-compact extra-dimensions can be envisaged [2], like in the five-dimensional model of Randall and Sundrum [3], where the bulk is endowed with a negative cosmological constant. Remarkably, in this model, the gravity felt by observers on the brane will behave, at least at first approximation, like usual four-dimensional gravity, because of the existence of zero-mode gravitons effectively confined in the brane.

When one considers a brane universe in the context of cosmology, it appears that one does not recover the standard Friedmann equations easily [4]. The reason is that the matter content of the brane enters quadratically in the equations governing the dynamics of the brane geometry. However, when one generalizes the idea of Randall and Sundrum to cosmology, i.e. when one assumes the existence of a negative cosmological constant in the bulk, adjusted so as to compensate the tension of the brane, one ends up with a cosmological evolution that is conventional at sufficiently low energies ( [5] and [6]).

The question now is whether this conventional behaviour will still be recovered in brane cosmology, when one relaxes the assumption of homogeneity and isotropy in the brane, in other words when one considers *inhomogeneous brane cosmology*. Up to now, almost all works (see e.g. [7]) on the cosmological aspects of the brane scenario have dealt with a homogeneous and isotropic brane universe. However, it is well known that a lot of information is contained in the cosmological perturbations of our Universe, and it is therefore of the uppermost importance to analyse the behaviour of cosmological perturbations in the context of brane cosmology. It would be crucial to test whether brane cosmology is compatible with current observations of cosmological perturbations, either the anisotropies of the Cosmic Microwave Background or the large scale structure data. It would also be interesting to check if the usual mechanism of generation of cosmological perturbations via amplification of quantum fluctuations during an inflationary phase would still be valid in brane cosmology (see [8]).

Another motivation to study the perturbations in the cosmological context is the question of the stabilisation of the radion and its implication on the cosmological evolution in the brane [9]. As shown by [10] in a non-cosmological context, a perturbative approach may be necessary for a full understanding of the radion.

The purpose of the present paper is to develop a formalism that can describe the evolution of the cosmological perturbations in the context of brane cosmology. In order to do so, we will proceed in several steps. First, one must compute the perturbations of the *bulk* Einstein's equations, which relate the perturbations of the metric to the perturbations of the bulk matter, if there is any. For this first step, we have computed the perturbed Einstein's equations in the more general case of a warped spacetime with any number of dimensions. A similar calculation has been done recently [11] in the case of maximally symmetric spacetimes.

The results valid for any dimension can be applied to the model we wish to focus on: a five-dimensional spacetime. It is then useful to distinguish several types of cosmological modes, following the familiar decomposition of the standard theory of cosmological perturbations (see e.g. [12] or [13]).

At this stage, however, the perturbations of the *matter in the brane* have not yet been taken into account. They play a rôle *only in the junctions conditions*: the jump in the derivatives (with respect to the fifth dimension) of the bulk metric perturbations is indeed governed by the matter perturbations in the brane. This is in this very unusual way that matter perturbations in our apparent Universe, i.e. the three-brane, are connected with the geometry perturbations in our Universe, i.e. simply the particular value on the brane of the bulk metric perturbations. It is nevertheless interesting to notice that the present problem has some analogy with the question of the interaction between a domain wall and gravitational waves in a four-dimensional spacetime [14].

The plan of the paper will be the following. In Section 2, we obtain the perturbations of Einstein's equation in the bulk, in the case of a general warped spacetime. In Section 3, we consider the five-dimensional model and recall what is known about the homogeneous and isotropic solutions in an empty bulk or with a negative cosmological constant. Section 4 is devoted to the bulk perturbations of the five-dimensional spacetime, whereas section 5 deals with the junction conditions that take into account the matter in the brane. Finally, we conclude in the last section.

## II. BULK PERTURBATIONS IN GENERAL WARPED SPACETIME

Although we will be ultimately interested in the perturbations of a five-dimensional spacetime, it is instructive to begin with a more general situation and to compute the perturbations of the Ricci tensor for a D-dimensional spacetime that can be considered as a warped product of a p-dimensional spacetime with a d-dimensional space. The metric has thus the particular form

$$\bar{g}_{AB}dx^A dx^B = \tilde{\gamma}_{ab}dx^a dx^b + a^2\{x^c\}\gamma_{ij}dx^i dx^j, \tag{1}$$

where $A, B, ...$ denote global spacetime indices, $a, b, ...$ indices of the p-dimensional spacetime with metric $\tilde{\gamma}_{ab}$ , and $i, j, ...$ indices of the d-dimensional space with metric $\gamma_{ij}$.

Let us denote $D_A$ the global covariant derivative associated with the metric $\bar{g}_{AB}$, whereas $\tilde{\nabla}_a$ will stand for the covariant derivative associated with the metric $\tilde{\gamma}_{ab}$ and $\nabla_i$ for the covariant derivative associated with the metric $\gamma_{ij}$. The nonvanishing mixed Christoffel symbols (those with indices of both types $a$ and $i$) are

$$\Gamma^i_{aj} = \frac{\partial_a a}{a}\gamma^i_j, \quad \Gamma^a_{ij} = -(a\partial^a a)\gamma_{ij}. \tag{2}$$

All tensors components will be decomposed in several sets, depending on the number of p-indices and of d-indices. And one can then decompose the action of the global spacetime covariant derivative $D_A$ into the action of the covariant derivative $\tilde{\nabla}_a$ and that of the covariant derivative $\nabla_i$. For illustration, on a vector $V^A = (V^a, V^i)$, the action of the covariant derivative $D_A$ gives

$$\begin{aligned}
D_a V^b &= \tilde{\nabla}_a V^b, \\
D_a V^i &= \frac{1}{a}\tilde{\nabla}_a(aV^i), \\
D_i V^a &= \nabla_i V^a - (a\partial^a a)\gamma_{ij}V^j, \\
D_i V^j &= \nabla_i V^j + \frac{\partial_b a}{a}\gamma^i_j V^b.
\end{aligned} \tag{3}$$

This type of formulas can be generalized to any type of tensor by using the mixed Christoffel symbols given above.

Let us now consider a linear perturbation of the spacetime metric, so that the total metric reads

$$ds^2 = (\bar{g}_{AB} + h_{AB})dx^A dx^B \equiv g_{AB}dx^A dx^B. \tag{4}$$

Our purpose will now be to compute the linear perturbation of the Ricci tensor, $\delta R_{AB}$. Quite generally, the expression of $\delta R_{AB}$ in terms of the metric perturbations is found to be of the following form (see e.g. [15])

$$\delta R_{AB} = -\frac{1}{2}D_A D_B h - \frac{1}{2}D^C D_C h_{AB} + D^C D_{(B}h_{A)C}, \tag{5}$$

where $h$ denotes the trace of the perturbation $h_{AB}$, namely

$$h \equiv \bar{g}^{AB}h_{AB}. \tag{6}$$

After some tedious but straightforward calculations, it is possible to express the various components of the linearized Ricci tensor more specifically in terms of the components $h_{ab}$, $h_{ai}$ and $h_{ij}$, and of the covariant derivatives $\tilde{\nabla}_a$ and $\nabla_i$. One finds

$$
\begin{aligned}
\delta R_{ab} = & -\frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}_b h - \frac{1}{2}\tilde{\nabla}^c\tilde{\nabla}_c h_{ab} + \tilde{\nabla}^c\tilde{\nabla}_{(a}h_{b)c} - \frac{1}{2}a^{-2}\nabla^2 h_{ab} + a^{-2}\nabla^k\tilde{\nabla}_{(a}h_{b)k} \\
& + \frac{d}{2}\frac{\tilde{\nabla}^c a}{a}\left(\tilde{\nabla}_a h_{bc} + \tilde{\nabla}_b h_{ac} - \tilde{\nabla}_c h_{ab}\right) - a^{-1}\tilde{\nabla}_{(a}a\tilde{\nabla}_{b)}\hat{h}, \qquad (7)\\
\delta R_{ai} = & -\frac{1}{2}\tilde{\nabla}_a\nabla_i h - \frac{1}{2}\tilde{\nabla}^b\tilde{\nabla}_b h_{ai} + \frac{1}{2}\tilde{\nabla}^b\tilde{\nabla}_a h_{bi} + \frac{1}{2}\nabla_i\tilde{\nabla}^b h_{ab} + \frac{1}{2}a^{-2}\tilde{\nabla}_a\nabla^k h_{ik} - \frac{1}{2}a^{-2}\nabla^k\nabla_k h_{ai} \\
& + \frac{1}{2}a^{-2}\nabla^k\nabla_i h_{ak} + \frac{1}{2}\frac{\tilde{\nabla}_a a}{a}\nabla_i(h-\hat{h}) - \frac{\tilde{\nabla}_a a}{a}\tilde{\nabla}^b h_{bi} + \frac{d}{2}\frac{\tilde{\nabla}^b a}{a}\tilde{\nabla}_a h_{bi} + \left(1-\frac{d}{2}\right)\frac{\tilde{\nabla}^b a}{a}\tilde{\nabla}_b h_{ai} \\
& - \left(1-\frac{d}{2}\right)\frac{\tilde{\nabla}^b a}{a}\nabla_i h_{ab} - \frac{\tilde{\nabla}_a}{a^3}\nabla^k h_{ik} - (d-1)\frac{\tilde{\nabla}^b a\tilde{\nabla}_a a}{a^2}h_{bi} - \frac{\tilde{\nabla}^b\tilde{\nabla}_a a}{a}h_{bi} \qquad (8)\\
\delta R_{ij} = & -\frac{1}{2}\nabla_i\nabla_j h - \frac{1}{2}\left(a\tilde{\nabla}^a a\right)\gamma_{ij}\tilde{\nabla}_a h - \frac{1}{2}a^{-2}\nabla^k\nabla_k h_{ij} + \frac{1}{2}a^{-2}\nabla^k\nabla_i h_{jk} + \frac{1}{2}a^{-2}\nabla^k\nabla_j h_{ik} \\
& - \frac{1}{2}\tilde{\nabla}^a\tilde{\nabla}_a h_{ij} + \frac{1}{2}\tilde{\nabla}^a\nabla_i h_{aj} + \frac{1}{2}\tilde{\nabla}^a\nabla_j h_{ai} + \frac{\tilde{\nabla}^a a}{a}\gamma_{ij}\nabla^k h_{ak} + \left(a\tilde{\nabla}^b a\right)\gamma_{ij}\tilde{\nabla}^a h_{ab} \\
& + \left(2-\frac{d}{2}\right)\frac{\tilde{\nabla}^a a}{a}\tilde{\nabla}_a h_{ij} + \left(\frac{d}{2}-1\right)\frac{\tilde{\nabla}^b a}{a}\nabla_i h_{bj} + \left(\frac{d}{2}-1\right)\frac{\tilde{\nabla}^b a}{a}\nabla_j h_{bi} \\
& + (d-1)(\tilde{\nabla}^a a\tilde{\nabla}^b a)\gamma_{ij}h_{ab} + a(\tilde{\nabla}^a\tilde{\nabla}^b a)\gamma_{ij}h_{ab} - 2\frac{\tilde{\nabla}^b a\tilde{\nabla}_b a}{a^2}h_{ij}, \qquad (9)
\end{aligned}
$$

with $\hat{h} \equiv a^{-2}\gamma^{ij}h_{ij}$.

*Gauge transformations.*

It is important to notice that perturbations which differ quantitatively can in fact describe the same geometry simply because they correspond to different systems of coordinates. Gauge transformations, corresponding to infinitesimal changes of coordinates

$$x^A \to x^A + \xi^A, \qquad (10)$$

induce the following transformations for the metric perturbations,

$$h_{AB} \to h_{AB} + D_A\xi_B + D_B\xi_A. \qquad (11)$$

This general expression gives for the three types of metric components or, decomposing into the two subsystems of coordinates,

$$
\begin{aligned}
h_{ab} &\to h_{ab} + \tilde{\gamma}_{bc}\tilde{\nabla}_a\xi^c + \tilde{\gamma}_{ac}\tilde{\nabla}_b\xi^c, \\
h_{ij} &\to h_{ij} + a^2\left(\gamma_{jk}\nabla_i\xi^k + \gamma_{ik}\nabla_j\xi^k\right) + 2a(\partial_a a)\gamma_{ij}\xi^a, \\
h_{ia} &\to h_{ia} + \tilde{\gamma}_{ab}\nabla_i\xi^b + a^2\gamma_{ij}\tilde{\nabla}_a\xi^j. \qquad (12)
\end{aligned}
$$

## III. FIVE-DIMENSIONAL BACKGROUND SPACETIME

In this section, we recall briefly the cosmological solutions that have been found in the case of five-dimensional spacetimes, with a metric of the form

$$ds^2 = -n^2(\tau, y)d\tau^2 + a^2(\tau, y)\gamma_{ij}dx^i dx^j + b^2(\tau, y)dy^2, \tag{13}$$

where the spatial three-surfaces, defined by $\tau$ and $y$ constant, are homogeneous and isotropic and $\gamma_{ij}$ is a maximally symmetric 3-dimensional metric ($k = -1, 0, 1$ will parametrize the spatial curvature).

The five-dimensional Einstein equations take the usual form

$$G_{AB} \equiv R_{AB} - \frac{1}{2}Rg_{AB} = \kappa^2 T_{AB}, \tag{14}$$

where $T_{AB}$ is the five-dimensional energy-momentum tensor.

With the above metric, the non-vanishing components of the Einstein tensor $G_{AB}$ are found to be

$$G_{00} = 3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) - \frac{n^2}{b^2}\left(\frac{a''}{a} + \frac{a'}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right)\right) + k\frac{n^2}{a^2}\right\}, \tag{15}$$

$$\begin{aligned} G_{ij} = {} & \frac{a^2}{b^2}\gamma_{ij}\left\{\frac{a'}{a}\left(\frac{a'}{a} + 2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n} + 2\frac{a'}{a}\right) + 2\frac{a''}{a} + \frac{n''}{n}\right\} \\ & + \frac{a^2}{n^2}\gamma_{ij}\left\{\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a} + 2\frac{\dot{n}}{n}\right) - 2\frac{\ddot{a}}{a} + \frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a} + \frac{\dot{n}}{n}\right) - \frac{\ddot{b}}{b}\right\} - k\gamma_{ij}, \end{aligned} \tag{16}$$

$$G_{05} = 3\left(\frac{n'}{n}\frac{\dot{a}}{a} + \frac{a'}{a}\frac{\dot{b}}{b} - \frac{\dot{a}'}{a}\right), \tag{17}$$

$$G_{55} = 3\left\{\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{b^2}{n^2}\left(\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n}\right) + \frac{\ddot{a}}{a}\right) - k\frac{b^2}{a^2}\right\}, \tag{18}$$

where a prime stands for a derivative with respect to $y$, and a dot for a derivative with respect to $\tau$. The stress-energy-momentum tensor can be decomposed into two parts,

$$T^A{}_B = \check{T}^A{}_B\big|_{\rm bulk} + T^A{}_B\big|_{\rm brane}, \tag{19}$$

where $\check{T}^A{}_B\big|_{\rm bulk}$ is the energy momentum tensor of the bulk matter and $T^A{}_B\big|_{\rm brane}$ corresponds to the matter content in the brane ($y = 0$). Since we consider in this section only strictly homogeneous and isotropic geometries inside the brane, the latter will be necessary of the form

$$T^A{}_B\big|_{\rm brane} = \frac{\delta(y)}{b}{\rm diag}\left(-\rho, p, p, p, 0\right), \tag{20}$$

where the energy density $\rho$ and pressure $p$ are functions only of time.

In the case of a general stress-energy tensor for the matter on the brane, and assuming a time-independent metric along the fifth dimension, explicit solutions for the whole metric have been found in two simple cases: the case where the bulk is empty [4] and the case where the bulk contains a cosmological constant [6], i.e.

$$\check{T}^A{}_B\big|_{\rm bulk} = {\rm diag}\left(-\rho_B, -\rho_B, -\rho_B, -\rho_B, -\rho_B\right), \tag{21}$$

with $\rho_B = const.$

In the case of a negative bulk cosmological constant, the most realistic, the explicit expressions for the metric components (with $b = 1$ and $n(t, y = 0) = 1$) are given by [6]

$$a(t,y) = \left\{ \frac{1}{2}\left(1 + \frac{\kappa^2\rho^2}{6\rho_B}\right) a_0^2 + \frac{3\mathcal{C}}{\kappa^2\rho_B a_0^2} + \left[\frac{1}{2}\left(1 - \frac{\kappa^2\rho^2}{6\rho_B}\right) a_0^2 - \frac{3\mathcal{C}}{\kappa^2\rho_B a_0^2}\right] \cosh(\mu y) \right.$$
$$\left. - \frac{\kappa\rho}{\sqrt{-6\rho_B}} a_0^2 \sinh(\mu|y|) \right\}^{1/2}, \tag{22}$$

with $\mu = \sqrt{-\frac{2\kappa^2}{3}\rho_B}$, and

$$n(t,y) = \frac{\dot{a}(t,y)}{\dot{a}_0(t)}, \tag{23}$$

where $a_0$ is the scale factor in the brane (i.e. in $y = 0$). The two functions $a_0(t)$ and $\rho(t)$, which appear in the above solution, are obtained by solving

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{\kappa^2}{6}\rho_B + \frac{\kappa^4}{36}\rho^2 + \frac{\mathcal{C}}{a_0^4} - \frac{k}{a_0^2}, \tag{24}$$

where $\mathcal{C}$ is a constant (of integration), and

$$\dot{\rho} + 3\frac{\dot{a}_0}{a_0}(\rho + p) = 0. \tag{25}$$

The last equation corresponds to the ordinary energy conservation law. But (24), analogous to the first Friedman equation, does not give the usual cosmological evolution because of the quadratic term $\rho^2$. It is however possible to recover the standard cosmological evolution, at least at sufficiently late times, if one decomposes $\rho$ into a tension $\sigma$ and an energy density of ordinary matter living in the brane and if one assumes a negative cosmological constant in the bulk that will compensate the $\sigma^2$ term (see [6]).

## IV. PERTURBATIONS OF A FIVE-DIMENSIONAL SPACETIME

After having described our reference spacetime in the previous section, we will now study linear perturbations about it. For this task, the calculations of Section 2, specialized to the case of five-dimensional spacetime, are going to be very useful. As it is clear from the previous section, the metric $\tilde{\gamma}_{ab}$ will be defined by

$$\tilde{\gamma}_{ab}dx^a dx^b = -n(t,y)^2 dt^2 + b(t,y)^2 dy^2. \tag{26}$$

The second metric $\gamma_{ij}$ is simply the metric covering the three ordinary spatial dimensions. In a cosmological context, there are three choices for $\gamma_{ij}$ depending on the spatial curvature of spacetime. For simplicity, it will be assumed that our Universe is spatially flat, which means that

$$\gamma_{ij} = \delta_{ij}. \tag{27}$$

From now on, we also choose to work in a Gaussian normal (GN) system of coordinates adapted to the three-brane, in which the brane is localized in $y = 0$ and the metric has the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{28}$$

where the greek indices refer to ordinary spacetime coordinates, i.e. the time coordinate $\tau$ and the three ordinary spatial dimensions $x^i$. For the background, this choice of gauge corresponds simply to set

$$b(t, y) = 1. \tag{29}$$

In the case of the perturbed spacetime, described by the metric (4), the choice of a GN system of coordinates will impose

$$h_{55} = h_{5\mu} = 0. \tag{30}$$

Taking this into account, the linearized metric (4) can now be written, quite generally, in the form

$$ds^2 = -n^2(1 + 2A)dt^2 + 2B_i dx^i dt + a^2 \left(\delta_{ij} + \hat{h}_{ij}\right) dx^i dx^j + dy^2. \tag{31}$$

Following Bardeen [16] (see also [12]), the linearized quantities specified above can be decomposed further, into so-called scalar, vector and tensor quantities, according to the expressions,

$$B_i = \partial_i B + \bar{B}_i, \tag{32}$$

where $\bar{B}_i$ satisfies $\partial_i \bar{B}^i = 0$, and

$$\hat{h}_{ij} = 2C\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i} E_{j)} + E_{ij}, \tag{33}$$

where $E_{ij}$ is transverse traceless, i.e. $\partial_i E^{ij} = 0$, $E^i_i = 0$. The quantities $A$, $C$, $B$ and $E$ are usually refered to as 'scalar' perturbations, $\bar{B}_i$ and $E_i$ as 'vector' perturbations, and $E_{ij}$ as 'tensor' perturbations.

### A. Gauge transformation

As explained earlier, the perturbations defined above can be quantitatively different but describe the same geometry if they are written in different coordinate systems. In order to distinguish gauge effects and physical degrees of freedom, it is useful to write down the effect of a coordinate change (or gauge transformation) on all the metric perturbations defined above. These transformations follow directly from the general expressions given in (12).

We will parametrize the infinitesimal coordinate transformation by the vector $\xi^A = (\xi^0, \xi^i, \xi^5)$. Let us first consider the components $h_{55}$, $h_{05}$ and $h_{i5}$, which vanish in a GN coordinate system. They transform according to the laws,

$$\begin{aligned}
h_{55} &\to h_{55} + 2\xi^{5\prime}, \\
h_{05} &\to h_{05} + \dot{\xi}^5 - n^2 \xi^{0\prime}, \\
h_{i5} &\to h_{i5} + \partial_i \xi^5 + a^2 \delta_{ij} \xi^{j\prime}.
\end{aligned} \tag{34}$$

In order to bring any coordinate system into a GN coordinate system, it is clear from the above relations that once one has used $\xi^5$ to adjust the position of the brane at $y = 0$, appropriate choices of $\xi^0$ and of $\xi^i$ will be required in order to make $h_{05}$ and $h_{i5}$ vanish. Note however that the GN gauge adapted to the brane is not completely fixed: there is some residual gauge freedom associated with parameters $\xi^0$ and $\xi^i$ that depend only on the four ordinary spacetime coordinates. This residual gauge freedom can be interpreted as possible redefinitions of the coodinates inside the brane worldsheet.

Let us now consider the transformations for the other components of the metric perturbations. To do that, it is convenient to decompose the spatial vector $x^i$ into

$$\xi^i = \partial^i \xi + \bar{\xi}^i \tag{35}$$

(where $\partial^i = \delta^{ij}\partial_j$), such that $\bar{\xi}^i$ is transverse, i.e. $\partial_i \bar{\xi}^i = 0$. With this decomposition, one can see that the three scalar parameters $\xi^0$, $\xi$ and $\xi^5$ will induce transformations in the subset of scalar perturbations, whereas $\bar{\xi}^i$ will act in the 'vector' subspace. The tensor perturbations $\bar{E}_{ij}$ will be left untouched by the gauge transformations. Specializing the expressions (12) to our particular case here, one finds the following transformations:

### *1. Scalar gauge transformations*

$$\begin{aligned}
A &\to A + \dot{\xi}^0 + \frac{\dot{n}}{n}\xi^0 + \frac{n'}{n}\xi^5, \\
B &\to B - n^2 \xi^0 + a^2 \dot{\xi}, \\
C &\to C + \frac{\dot{a}}{a}\xi^0 + \frac{a'}{a}\xi^5, \\
E &\to E + \xi,
\end{aligned} \tag{36}$$

### *2. Vector gauge transformations*

$$\begin{aligned}
\bar{B}_i &\to \bar{B}_i + a^2 \dot{\bar{\xi}}_i, \\
E_i &\to E_i + \bar{\xi}_i.
\end{aligned} \tag{37}$$

In the above transformations, we have considered the most general gauge transformation. But if one considers only gauge transformations inside the subset of GN coordinate systems, which will be the case in the following, then $\xi^5$ disappears from (36) and the parameters $\xi^0$, $\xi$ and $\bar{\xi}$ cannot depend on the fifth coordinate.

Finally let us remark that one can use the remaining gauge freedom within the GN subset to impose some additional gauge conditions. For example, for scalar quantities, a choice which is familiar in the standard theory of cosmological perturbations is to impose

$$B = 0, \quad E = 0, \tag{38}$$

which corresponds to the so-called longitudinal (or Newtonian) gauge. However, these conditions can be imposed only on one hypersurface, $y = 0$ say, but not everywhere in the bulk, because $B$ and $E$ a priori depend on the coordinate $y$.

### B. Perturbed Einstein equations in the bulk

Let us now compute the equations governing the perturbations in the bulk, ignoring for the moment the presence of the brane. The five dimensional Einstein's equations (14), in the bulk, can be rewritten

$$R_{AB} = \kappa^2 \left( \check{T}_{AB} - \frac{1}{3} \check{T} g_{AB} \right). \tag{39}$$

Therefore, the perturbed Einstein equations in the bulk have the form

$$\delta R_{AB} = \kappa^2 \left( \delta \check{T}_{AB} - \frac{1}{3} \check{T} h_{AB} - \frac{1}{3} g_{AB} \delta \check{T} \right). \tag{40}$$

Since the background solutions have been found explicity, as recalled in Section 3, the cases of an empty bulk or of a cosmological constant are of particular interest, but of course, the present formalism applies to any bulk energy-momentum tensor. In the case of an empty bulk, the perturbed Einstein's equations are simply

$$\delta R_{AB} = 0, \tag{41}$$

whereas in the case of a bulk with a cosmological constant $\Lambda = \kappa^2 \rho_B$, the perturbed Einstein's equations read

$$\delta R_{AB} = \frac{2}{3} \Lambda h_{AB}. \tag{42}$$

The remaining task consists in computing explicitly the components of the perturbed Ricci tensor. Using the expressions (7-9), one obtains the following expressions, which can conveniently be separated in scalar, vector and tensor parts.

#### 1. Scalar components

$$\begin{aligned}
\delta R^S_{00} = {} & n^2 \left[ A'' + \left( 3\frac{a'}{a} + 2\frac{n'}{n} \right) A' + 3\frac{n'}{n} C' + \left( 6\frac{a'n'}{an} + 2\frac{n''}{n} \right) A \right] \\
& -3\ddot{C} + 3\frac{\dot{a}}{a}\dot{A} + \left( 3\frac{\dot{n}}{n} - 6\frac{\dot{a}}{a} \right) \dot{C} + \frac{n^2}{a^2} \Delta A \\
& -\Delta \ddot{E} + \left( \frac{\dot{n}}{n} - 2\frac{\dot{a}}{a} \right) \Delta \dot{E} + n n' \Delta E' + a^{-2} \Delta \dot{B} - \frac{\dot{n}}{a^2 n} \Delta B,
\end{aligned} \tag{43}$$

$$\delta R^S_{0i} = \partial_i \left\{ -2\dot{C} + 2\frac{\dot{a}}{a} A - \frac{1}{2} B'' + \frac{1}{2} \left( \frac{n'}{n} - \frac{a'}{a} \right) B' + \left[ \frac{1}{n^2} \left( \frac{\ddot{a}}{a} - \frac{\dot{n}\dot{a}}{na} + 2\frac{\dot{a}^2}{a^2} \right) - 2\frac{a'n'}{an} \right] B \right\}, \tag{44}$$

$$\begin{aligned}
\delta R^S_{ij} = {} & -a^2 \left\{ \left( 4\frac{\dot{a}^2}{a^2 n^2} - 2\frac{\dot{a}\dot{n}}{a n^3} + 2\frac{\ddot{a}}{a n^2} \right) A + 2 \left( \frac{a'n'}{an} + 2\frac{a'^2}{a^2} + \frac{a''}{a} + \frac{\dot{a}\dot{n}}{a n^3} - 2\frac{\dot{a}^2}{a^2 n^2} - \frac{\ddot{a}}{a n^2} \right) C \right. \\
& + \frac{a'}{a} A' + \left( 6\frac{a'}{a} + \frac{n'}{n} \right) C' + \frac{\dot{a}}{a n^2} \dot{A} \\
& \left. + C'' + \left( \frac{\dot{n}}{n^3} - 6\frac{\dot{a}}{a n^2} \right) \dot{C} - \frac{1}{n^2} \ddot{C} + \frac{\dot{a}}{n^2 a^3} \Delta B - \frac{\dot{a}}{n^2 a} \Delta \dot{E} + \frac{a'}{a} \Delta E' \right\} \delta_{ij} - \Delta C \delta_{ij}
\end{aligned}$$

$$+\partial_i\partial_j\left[-A-C+\frac{a^2}{n^2}\ddot{E}+\left(\frac{3a\dot{a}}{n^2}-\frac{a^2\dot{n}}{n^3}\right)\dot{E}-a^2E''-\left(3aa'+\frac{n'a^2}{n}\right)E'\right.$$
$$\left.+2\left(\frac{a\ddot{a}}{n^2}+2\frac{\dot{a}^2}{n^2}-\frac{\dot{n}a\dot{a}}{n^3}-aa''-2a'^2-\frac{n'aa'}{n}\right)E-\frac{1}{n^2}\dot{B}+\left(\frac{\dot{n}}{n^3}-\frac{\dot{a}}{an^2}\right)B\right] \tag{45}$$
$$\delta R^S_{i5}=\partial_i\left[-A'-2C'+\left(\frac{a'}{a}-\frac{n'}{n}\right)A-\frac{1}{2n^2}\dot{B}'+\frac{a'}{an^2}\dot{B}+\left(\frac{\dot{n}}{2n^3}-\frac{3\dot{a}}{2n^2a}\right)B'\right.$$
$$\left.+\left(\frac{\dot{a}'}{an^2}+2\frac{\dot{a}a'}{n^2a^2}-\frac{\dot{n}a'}{n^3a}\right)B\right], \tag{46}$$
$$\delta R^S_{55}=-6\frac{a'}{a}C'-2\frac{n'}{n}A'-A''-3C''-\Delta E''-2\frac{a'}{a}\Delta E', \tag{47}$$
$$\delta R^S_{05}=3\left[\frac{n'}{n}\dot{C}-\dot{C}'+\frac{\dot{a}}{a}A'-\frac{\dot{a}}{a}C'-\frac{a'}{a}\dot{C}\right]+\frac{1}{2}a^{-2}\Delta B'-\frac{n'}{a^2n}\Delta B$$
$$-\Delta\dot{E}'+\left(\frac{n'}{n}-\frac{a'}{a}\right)\Delta\dot{E}-\frac{\dot{a}}{a}\Delta E'. \tag{48}$$

### 2. Vector components

$$\delta R^V_{0i}=\frac{1}{2}\Delta\dot{E}_i-\frac{1}{2}\bar{B}_i''+\frac{1}{2}\left(\frac{n'}{n}-\frac{a'}{a}\right)\bar{B}_i'+\left[\frac{1}{n^2}\left(\frac{\ddot{a}}{a}-\frac{\dot{n}\dot{a}}{na}+2\frac{\dot{a}^2}{a^2}\right)-2\frac{a'n'}{an}\right]\bar{B}_i-\frac{1}{2a^2}\Delta\bar{B}_i, \tag{49}$$
$$\delta R^V_{i5}=\frac{1}{2}\Delta E_i'-\frac{1}{2n^2}\dot{\bar{B}}_i'+\frac{a'}{an^2}\dot{\bar{B}}_i+\left(\frac{\dot{n}}{2n^3}-\frac{3\dot{a}}{2n^2a}\right)\bar{B}_i'+\left(\frac{\dot{a}'}{an^2}+2\frac{\dot{a}a'}{n^2a^2}-\frac{\dot{n}a'}{n^3a}\right)\bar{B}_i \tag{50}$$
$$\delta R^V_{ij}=\frac{a^2}{n^2}\partial_{(i}\ddot{E}_{j)}+\left(\frac{3a\dot{a}}{n^2}-\frac{a^2\dot{n}}{n^3}\right)\partial_{(i}\dot{E}_{j)}-a^2\partial_{(i}E_{j)}''-\left(3aa'+\frac{n'a^2}{n}\right)\partial_{(i}E_{j)}'$$
$$+2\left(\frac{a\ddot{a}}{n^2}+2\frac{\dot{a}^2}{n^2}-\frac{\dot{n}a\dot{a}}{n^3}-aa''-2a'^2-\frac{n'aa'}{n}\right)\partial_{(i}E_{j)}$$
$$-\frac{1}{n^2}\partial_{(i}\dot{\bar{B}}_{j)}+\left(\frac{\dot{n}}{n^3}-\frac{\dot{a}}{an^2}\right)\partial_{(i}\bar{B}_{j)} \tag{51}$$

### 3. Tensor components

$$\delta R^T_{ij}=-\frac{1}{2}\Delta E_{ij}+\frac{a^2}{2n^2}\ddot{E}_{ij}+\left(\frac{3a\dot{a}}{2n^2}-\frac{a^2\dot{n}}{2n^3}\right)\dot{E}_{ij}-\frac{1}{2}a^2E_{ij}''-\left(\frac{3}{2}aa'+\frac{n'a^2}{2n}\right)E_{ij}'$$
$$+\left(\frac{a\ddot{a}}{n^2}+2\frac{\dot{a}^2}{n^2}-\frac{\dot{n}a\dot{a}}{n^3}-aa''-2a'^2-\frac{n'aa'}{n}\right)E_{ij}. \tag{52}$$

## V. JUNCTION CONDITIONS

In the previous section, we have considered only the perturbations in the bulk. It is now time to take into account the brane itself, and in particular the perturbations of the matter in the brane. The connection between the metric perturbations, living in the bulk, and the matter perturbations confined to the brane is made via the junction conditions [17].

The energy-momentum tensor describing matter content of the brane will be written in the form

$$T^{\mu}{}_{\nu}|_{\rm brane} = \delta(y) S^{\mu}_{\nu}, \tag{53}$$

and, in the background, it has the perfect fluid form

$$S^{\mu}_{\nu} = (\rho + P) u^{\mu} u_{\nu} + P g^{\mu}_{\nu}. \tag{54}$$

The junction conditions are found to be given, in the five-dimensional case [4], by

$$[K_{\mu\nu}] = -\kappa^2 \left( S_{\mu\nu} - \frac{1}{3} S g_{\mu\nu} \right), \tag{55}$$

where the brackets denote the jump across the brane, of the extrinsic curvature $K_{\mu\nu}$ and $S \equiv S^{\mu}_{\mu}$. In a Gaussian normal coordinate system, the extrinsic curvature is given by the simple expression

$$K_{\mu\nu} = \frac{1}{2} \partial_y g_{\mu\nu}. \tag{56}$$

In the present context, it is convenient to decompose the junction conditions (55) into a background part and a perturbed part. The use of the background junction conditions is necessary to find solutions of the Einstein equations when homogeneity and isotropy in the brane are assumed, such as those given in (22-23). The junction conditions for the background have been given in [4]

$$\frac{[a']}{a_0 b_0} = -\frac{\kappa^2}{3} \rho, \tag{57}$$

$$\frac{[n']}{n_0 b_0} = \frac{\kappa^2}{3} (3p + 2\rho), \tag{58}$$

where the subscript 0 for $a, b, n$ means that these functions are taken in $y = 0$.

Let us now consider the junction conditions for the perturbations, which can be written

$$[\delta K_{\mu\nu}] = \kappa^2 \left( -\delta S_{\mu\nu} + \frac{1}{3} g_{\mu\nu} \delta S + \frac{1}{3} S h_{\mu\nu} \right), \tag{59}$$

Using the expression (56) for the extrinsic curvature in the GN gauge as well as the $y \to -y$ symmetry, one ends up with the following condition,

$$h'_{\mu\nu|y=0^+} = \kappa^2 \left( -\delta S_{\mu\nu} + \frac{1}{3} g_{\mu\nu} \delta S + \frac{1}{3} S h_{\mu\nu} \right). \tag{60}$$

It is then useful to decompose further these junction conditions by distinguishing space and time. Let us begin with the perturbations of the energy-momentum tensor in the brane. The perturbations for the unit four-velocity can be written

$$\delta u^{\mu} = \left\{ -n^{-1} A, a^{-1} v^i \right\}, \tag{61}$$

where the expression for $\delta u^0$ follows automatically from the normalization condition satisfied by $u^\mu$. The perturbations of the energy-momentum tensor then have the following form:

$$\begin{aligned}
\delta S_{00} &= n^2\delta\rho + 2\rho n^2 A, \\
\delta S_{0i} &= -(\rho+P)nav_i - \rho B_i, \\
\delta S_{ij} &= a^2\delta P\delta_{ij} + Ph_{ij} + a^2\pi_{ij},
\end{aligned} \tag{62}$$

with $v_i \equiv \delta_{ij}v^j$ and where $\pi_{ij}$ is the anisotropic stress tensor. Once more it is possible to decompose the above expressions for the brane matter energy-momentum tensor into scalar, vector and tensor components, using

$$v_i = \partial_i v + \bar{v}_i, \tag{63}$$

with $\partial_i\bar{v}^i = 0$ and

$$\pi_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)\pi + 2\partial_{(i}\pi_{j)} + \bar{\pi}_{ij}, \tag{64}$$

with as usual $\pi_i$ transverse and $\bar{\pi}_{ij}$ transverse traceless. Substituting the above expressions for the perturbed energy-momentum tensor and using the background junction conditions (57-58) yields the following conditions for the perturbations

$$A'_{|y=0^+} = \frac{\kappa^2}{6}\left(2\delta\rho + 3\delta P\right), \tag{65}$$

$$(n^{-2}B_i)'_{|y=0^+} = \kappa^2(\rho+P)\frac{a_0}{n_0}v_i, \tag{66}$$

$$\hat{h}'_{ij|y=0^+} = -\kappa^2\left(\frac{1}{3}\delta\rho\delta_{ij} + \pi_{ij}\right). \tag{67}$$

These conditions can also be decomposed, in a straightforward manner, into scalar, vector and tensor junction conditions.

## VI. CONCLUSION

In the present work, we have developed a formalism in order to deal with the evolution of cosmological perturbations in a brane universe. This formalism has the advantage to introduce quantities that are similar to the usual treatments of cosmological perturbations in standard cosmology, thus making easier in the future a comparison between brane cosmoloical perturbations and observable quantities, such as the large scale structure and the temperature anisotropies.

However, the equations governing the perturbations in the brane scenario are much more complicated than in standard cosmology. The main reason is that the equations for the metric perturbations, after a usual decomposition into ordinary (spatial) Fourier modes, will contain partial derivatives with respect to time and with respect to the fifth coordinate, in contrast with standard cosmology where one ends up with only ordinary differential equations with respect to time. Solving these equations appears to be quite a challenge.

Note: while the present work was being completed, two works on the same subject ( [18] and [19]) have appeared on hep-th.

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## REFERENCES


[1] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys.Lett. B429, (1998) 263; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. **B436**, 257 (1998); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. **D59** (1999) 086004.

[2] V.A.Rubakov, M.E.Shaposhnikov, Phys.Lett.B125,139 (1983); M. Visser, Phys.Lett. B159,22 (1985); E.J. Squires, Phys.Lett. B167, 286 (1986); G.W. Gibbons, D.L. Wiltshire, Nucl. Phys. **B287**, 717 (1987); M. Gogberashvili, hep-ph/9812296; hep-ph/9812365; hep-ph/9904383.

[3] L.Randall, R.Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [hep-th/9906064].

[4] P.Binétruy, C.Deffayet, D.Langlois, *Nucl. Phys.* **B 565**, 269 (2000) [hep-th/9905012].

[5] C. Csáki, M. Graesser, C. Kolda, J. Terning, Phys. Lett. **B462**, 34 (1999); J.M. Cline, C.Grosjean, G.Servant, *Phys. Rev. Lett.* **83**, 4245 (1999).

[6] P. Binétruy, C. Deffayet, U. Ellwanger, D. Langlois, *Phys.Lett.***B 477**, 285 (2000) [hep-th/9910219].

[7] A. Lukas, B.A. Ovrut, D. Waldram, hep-th/9902071; H.A. Chamblin, H.S. Reall, hep-th/9903225; N. Kaloper, *Phys. Rev. D* **60**, 123506 (1999); T. Nihei, *Phys. Lett. B* **465**, 81 (1999); H.B. Kim, H.D. Kim, *Phys. Rev. D* **61**, 064003 (2000); Flanagan, E., Tye, S. and Wasserman, I.: *hep-th/9910498* (1999). P.Kanti, I.I. Kogan, K.A. Olive, M.Pospelov, Phys. Lett. **B468**, 31 (1999);; P. Kraus, JHEP 9912 (1999) 011; Ida, D.: *gr-qc/9912002* (1999); T. Shiromizu, K. Maeda, M. Sasaki, gr-qc/9910076; N. Deruelle, T. Dolezel, gr-qc/0004021; J. Lesgourgues, S. Pastor, M. Peloso, L. Sorbo, hep-ph/0004086; H. Stoica, H. Tye, I. Wasserman, hep-th/0004126;

[8] R. Maartens, D. Wands, B. Bassett, I. Heard, hep-ph/9912464

[9] C. Csaki, M. Graesser, L. Randall, J. Terning, hep-ph/9911406

[10] C. Charmousis, R. Gregory, V. A. Rubakov, hep-th/9912160

[11] S. Mukohyama, hep-th/0004067, “Gauge invariant gravitational perturbations of maximally symmetric spacetimes”

[12] V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, *Phys. Reports* **215**, 203 (1992)

[13] H. Kodama, M. Sasaki, *Int. J. Mod. Phys.* **A 1**, 265 (1986)

[14] A. Ishibashi, H. Ishihara, *Phys. Rev.* **D 56**, 3446 (1997).

[15] R.M. Wald, “General Relativity”, The University of Chicago Press (1984).

[16] J.M. Bardeen, *Phys. Rev.* **D 22**, 1882 (1980)

[17] W. Israel, Nuovo Cimento B 44 (1966).

[18] H. Kodama, A. Ishibashi, O. Seto, hep-th/0004160 “Brane world cosmology: gauge invariant formalism for perturbations”

[19] R. Maartens, hep-th/0004166 “Cosmological dynamics on the brane”